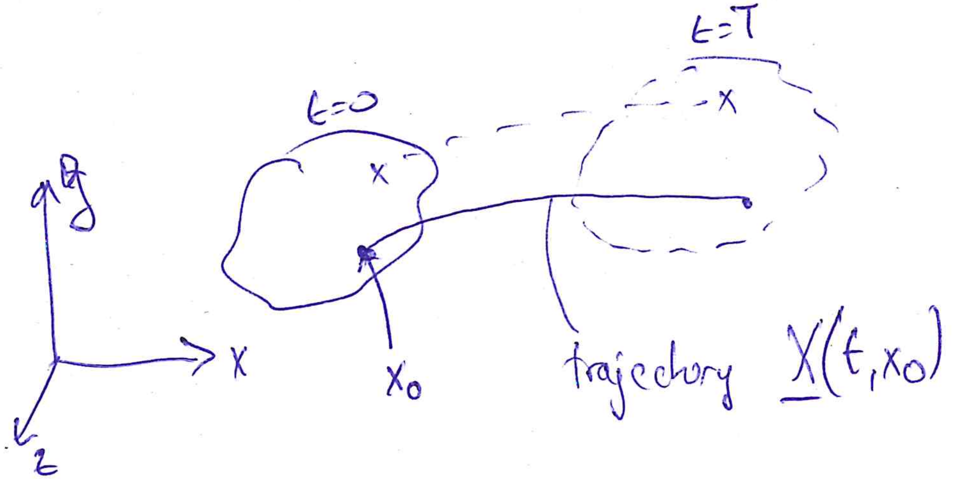
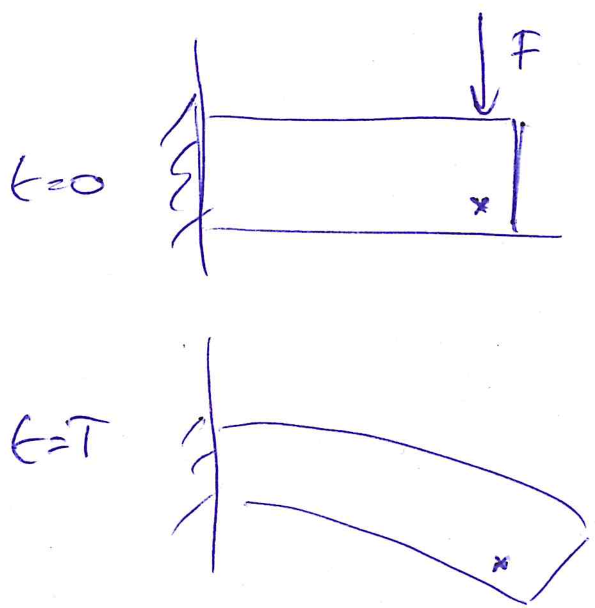
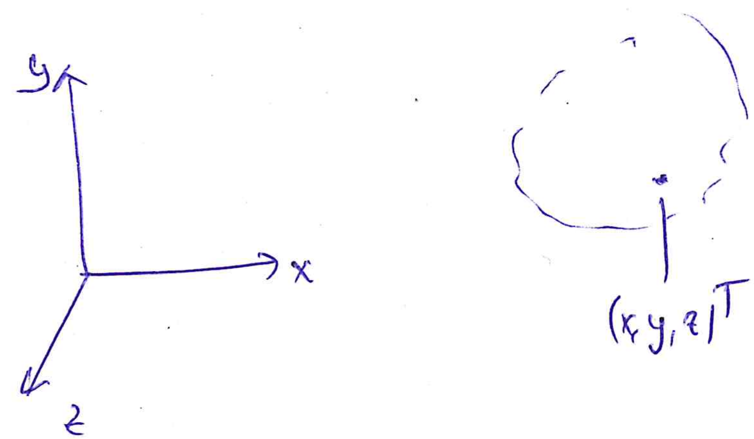
Lagrangian description





$$\begin{cases} \dfrac{dX(t)}{dt} = v \\ X(t, x_0) = x \end{cases}$$

Eulerian description



- we know $v(x,t)$

Ex 1.a) Eulerian to Lagrangian

(Ruderman, 2010)

$$X_1 = e^{t/t_0}(x_0)_1 \quad X_2 = \cosh(t/t_0)(x_0)_2 \quad (*)$$
$$X_3 = e^{-t/t_0}(x_0)_3$$

$X_i(t, x_0)$: trajectory in Lag.

$(x_0)_i$: marked position

$$\phi(X_i(t), t) = (x_0)_1^2 + (x_0)_2^2 + (x_0)_3^2$$

1. rewrite in Eul. variables!

a) Invert variables X_i

$$(x_0)_1 = e^{-t/t_0} X_1 \quad \& \quad (x_0)_2 = \frac{1}{\cosh(t/t_0)} X_2$$
$$\& \quad (x_0)_3 = e^{t/t_0} X_3$$

→ go to Eulerian frame

$$X_i(t, x_0) = x_i \qquad x_i = (x, y, z)^T$$

$$(x_0)_1 = e^{-t/t_0} x$$
← point in space in Eul. descr.

$(x_0)_2 = \ldots$

$(x_0)_3 = \ldots$

b) ~~Subst~~ Substitution into ϕ

$$\phi(x, y, z, t) = e^{-2t/t_0} x^2 + \frac{y^2}{\cosh^2(t/t_0)} + e^{2t/t_0} z^2 \qquad (**)$$

b) Compute

$$\frac{d\phi}{dt}, \quad \frac{D\phi}{Dt}$$

by using their definitions

$$\left[\frac{d\phi(X(t),t)}{dt} = \frac{D\phi(t,x,y,z)}{Dt} \right]$$

$$\frac{d\phi}{dt} = \frac{d}{dt}\left((x_0)_x^2 + (x_0)_z^2 + (x_0)_y^2 \right) = 0$$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + v_i \frac{\partial\phi}{\partial x_i} = 0$$

a) $$\frac{\partial\phi}{\partial t} = \frac{\partial\phi(x,y,z,t)}{\partial t} \overset{(**)}{=} \dots$$

$$v_i = \frac{dX(t)}{dt} \overset{(*)}{=}$$

$$\frac{\partial\phi}{\partial x_i} \underset{(**)}{=} \dots$$

Ex. 2 Eulerian to Lagrangian

(Ruderman, 2010)

Given

Eulerian description

$$v(t,x,y,z) = \begin{pmatrix} x/(t+t_0) \\ y/t_0 \\ z/t_0 \tanh(t/t_0) \end{pmatrix}$$

$$\varphi(t,x,y,z) = e^{-t/t_0}\; xyz$$

1. Find the trajectories ~~of~~ $X(t)$:

$$v_i = \frac{dX(t)}{dt} \quad ; \quad \text{initial condition } X_i(t=0) = (x_0)_i$$

$$x_i = X_i(t) \quad \frac{dX_1}{dt} = \frac{x}{t+t_0} \quad \cancel{\frac{dX_1}{dt}} \Rightarrow X_1 = (x_0)_1 (1 + t/t_0)$$

$$\Leftrightarrow \boxed{\frac{dX_1}{dt} = \frac{X_1}{t+t_0}}$$

$$\frac{dX_2}{dt} = \frac{y}{t_0} \underset{\text{lagr. frame}}{\Leftrightarrow} \frac{dX_2}{dt} = \frac{X_2}{t_0}$$

$$\rightsquigarrow X_2 = (x_0)_2\, e^{t/t_0}$$

$$\frac{dX_3}{dt} = \frac{z}{t_0} \tanh(t/t_0) \Leftrightarrow \frac{dX_3}{dt} = \frac{X_3}{t_0} \tanh(t/t_0)$$

$$\rightsquigarrow \cancel{X_3(} \; X_3 = (x_0)_3 \cosh(t/t_0)$$

2. Substitute $X_i(t)$ into φ

$$\varphi(X_1, X_2, X_3, t) = (x_0)_1 (x_0)_2 (x_0)_3 \underbrace{(1+t/t_0)\cosh(t/t_0)}$$